Field transformations and Symmetries

Noether's Theorem, Schwinger-Dyson eqns, and Ward-Takahashi identities

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1   The standard approach

The analysis of field transformations that is traditionally presented in textbooks (e.g. Srednicki, "Quantum Field Theory", Sec. 22) has two significant limitations.
(i) It only applies to field theories whose Lagrange density $\mathcal{L}(\varphi, \partial_\mu \varphi)$ is just a function of the field φ and its first derivative $\partial_\mu \varphi$: the proof fails if higher derivatives are present.
(ii) It cannot be applied to field theories that are regulated by formulating them on a space-time lattice, since in those theories a derivative is discretized to something like $(\varphi(z + a) - \varphi(z - a))/2a$, so the action is a function of the field values $\varphi(x)$ at lattice sites, and $\mathcal{L}(x)$ just depends on $\varphi(x)$. In the continuum limit, where the lattic spacing goes to zero, this is still true, and there is never any separate dependence of $\mathcal{L}(x)$ on $\partial_\mu \varphi$.

Here we give a much more general analysis, applicable to any field theory, although we assume the fields are scalars. We start by obtaining a general expression relating the change in the Lagrange density under a field transformation to the change in the action. This can then be used to derive Noether's theorem, and also, as described in Srednicki's textbook, the Schwinger-Dyson equations and the Ward-Takahashi identities.

The proof proceeds by manipulation of the functional integral over field histories, and we will use integration by parts to move derivatives from one factor to another, so we will assume that all our field configurations are "well-behaved", meaning that all fields go to zero sufficiently quickly at space-time infinity that there are no boundary terms. Thus "well-behaved" field configurations f and g obey

$$\int f(x) \partial_\mu g(x)\, d^d x = - \int (\partial_\mu f(x))\, g(x)\, d^d x \tag{1}$$

2   Infinitesimal field transformations

Consider some infinitesimal transformation T of the fields, which may or may not be a symmetry:

$$T: \quad \varphi(x) \to \varphi(x) + \delta\epsilon \Delta\varphi(x) \tag{2}$$

The field $\varphi(x)$ might be a single scalar field, or a vector $\varphi_a(x)$ of fields. $\Delta\varphi$ is not infinitesimal, it gives the "shape" of the field transformation. Examples might

be $\Delta\varphi(x) = a^\mu \partial_\mu \varphi(x)$ (translation by a^μ) or $\Delta\varphi_a(x) = \varepsilon^{ab}\varphi_b(x)$ (rotating in the internal space of two scalar fields). $\delta\epsilon$ is the infinitesimal parameter that we will assume to be very small, so we work to lowest non-trivial order in $\delta\epsilon$.

2.1 Change in the Lagrange density

The change in the Lagrange density can be broken into a part that is a total derivative of a well-behaved vector field and a part that is not,

$$\delta\mathcal{L}(x) = \left(\partial_\mu K^\mu(x) + I_{(0)}(x)\right)\delta\epsilon \tag{3}$$

$I_{(0)}$ is defined by stipulating that it cannot be written as a total derivative, i.e. there is no non-zero well-behaved vector field E^μ such that $I_{(0)}(x) = \partial_\mu E^\mu(x)$. If $\delta\mathcal{L}$ is a total derivative then $I_{(0)}(x) = 0$; if $\delta\mathcal{L}$ is not just a total derivative then $I_{(0)}(x)$ is not uniquely defined (one can add divergences to $I_{(0)}$ and subtract them from $\partial_\mu K^\mu$) but it is definitely non-zero. The change in the action is

$$\delta S = \int \delta\mathcal{L}(x)\, d^d x \; . \tag{4}$$

Now let's think about symmetries. The definition of a symmetry transformation is that $\delta S = 0$, and we claim that for a symmetry transformation the Lagrange density changes by a total derivative, so $I_{(0)} = 0$:

$$\begin{array}{ccc}
T \text{ is a symmetry} & & I_{(0)}(x) = 0 \\
\text{i.e. } \delta S = 0 & \Leftrightarrow & \delta\mathcal{L}(x) = \partial_\mu K^\mu(x)\,\delta\epsilon
\end{array} \tag{5}$$

How do we prove (5)? It is obvious that $I_{(0)} = 0 \Rightarrow \delta S = 0$, since if $I_{(0)} = 0$ then $\delta\mathcal{L}$ is a total derivative of a well-behaved vector function, which integrates to zero. To show that $\delta S = 0 \Rightarrow I_{(0)} = 0$ we have to show that if $\delta\mathcal{L}(x)$ integrates to zero then it can be always written as a total derivative of a well-behaved vector field. This actually follows from basic electrostatics: if we think of $\delta\mathcal{L}(x)$ as a well-behaved charge distribution $\rho(\vec{x})$ with zero net charge then it can always be written as a total derivative of a well-behaved vector field, since there is a electric field $\vec{E}$ which obeys $\mathrm{div}\vec{E} = \rho$ and drops off to zero at infinity so rapidly that it yields no boundary terms on integration. This is true for electrostatics in any number of dimensions. So (5) is proved.

2.2 Position-dependent modulation of the field transformation

The slick way to obtain the Noether current corresponding to a given symmetry T is to gauge the transformation T, and find the current j_μ that couples to the gauge field. If the transformation is a symmetry then the current will be conserved.

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That's fine if you already know about gauge fields. Here we give an elementary derivation.

Generalize to a position-dependent transformation, by making $\delta\epsilon$ a function of position:

$$T_{\text{gauged}} : \varphi(x) \to \varphi(x) + \delta\epsilon(x)\Delta\varphi(x) \tag{6}$$

It may seem redundant to have x-dependence in $\Delta\varphi(x)$ and in $\delta\epsilon(x)$, but it isn't. The x-dependence of $\Delta\varphi(x)$ is fixed by our choice of transformation T, but for a given T, we can freely vary $\delta\epsilon(x)$ to be *any* function of x. Even if T is a symmetry, T_{gauged} will *not* be a symmetry, except in the special case where $\delta\epsilon(x)$ is constant.

The change in the Lagrange density under T_{gauged} now includes additional terms arising from the position dependence of $\delta\epsilon$. These are therefore derivatives of $\delta\epsilon$:

$$\delta\mathcal{L}(x) = \Big(I_{(0)}(x) + \partial_\mu K^\mu(x)\Big)\delta\epsilon(x) + I_{(1)}^\nu(x)\partial_\nu\delta\epsilon(x) + I_{(2)}^{\nu\rho}(x)\partial_\nu\partial_\rho\delta\epsilon(x) + \cdots \tag{7}$$

We integrate over space-time to obtain the change in the action. We can then use integration by parts to move all the derivatives off the $\delta\epsilon$ and on to the I coefficients, and obtain

$$\begin{aligned}
\delta S &= \int d^d x\, \delta\epsilon(x)\Big(I_{(0)}(x) + \partial_\mu K^\mu(x)\Big) - \delta\epsilon(x)\partial_\mu I_{(1)}^\mu(x) + \delta\epsilon(x)\partial_\mu\partial_\nu I_{(2)}^{\mu\nu}(x) + \cdots \\
&= \int d^d x\, \delta\epsilon(x)\Big(I_{(0)}(x) + \partial_\mu j^\mu(x)\Big)
\end{aligned} \tag{8}$$

where the Noether current is

$$j^\mu(x) = K^\mu(x) - I_{(1)}^\mu(x) + \partial_\nu I_{(2)}^{\mu\nu}(x) + \cdots \tag{9}$$

and $I_{(n)}(x)$ are defined by (7). Our final step is to express (8) more compactly as a functional derivative:

$$\boxed{\frac{\delta S}{\delta\epsilon(x)} = I_{(0)}(x) + \partial_\mu j^\mu(x)} \tag{10}$$

which is analogous to Srednicki (22.7). (10) is a powerful general theorem that underlies Noether's theorem, the Schwinger-Dyson equations, and the Ward-Takahashi identities. It subsumes the following special cases: (a) If the original transformation T was a symmetry, then, by (5), $I_{(0)}(x) = 0$. (b) If the field configuration is a solution to the equation of motion then it is a stationary point of the action under *any* infinitesimal change in the field configuration, so $\delta S/\delta\epsilon(x) = 0$.

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2.3   Noether's theorem

From (10) we see that if the original transformation T is a symmetry and the field is a solution to the equation of motion, then the Noether current is conserved:

$$\partial_\mu j^\mu = 0 \tag{11}$$

That is Noether's theorem. To explicitly obtain the Noether current you have to do the position-dependent field transformation (6) on the Lagrange density and obtain the change (7), then use (9) to obtain the current.